# Discrete multidimensional signals

## Notations and definitions

- Discrete-domain signal (square brakets): $a[\boldsymbol{k}]$ with $\boldsymbol{k} = (k_1, \cdots, k_d) \in \mathbb{Z}^d$
- $\ell_p$-norm: $\|a\|_{\ell_p(\mathbb{Z}^d)} = \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |a[\boldsymbol{k}]|^p\right)^{\frac{1}{p}}$, for $1 \le p < +\infty$
- $\ell_2$-inner product: $\langle a, b\rangle_{\ell_2(\mathbb{Z}^d)} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\, b^*[\boldsymbol{k}]$
- Signal energy: $\|a\|^2_{\ell_2(\mathbb{Z}^d)} \triangleq \langle a, a\rangle_{\ell_2(\mathbb{Z}^d)} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |a[\boldsymbol{k}]|^2$
- Space of finite-energy sequences: $\ell_2(\mathbb{Z}^d) = \left\{a[\boldsymbol{k}] : \boldsymbol{k} \in \mathbb{Z}^d, \|a\|^2_{\ell_2(\mathbb{Z}^d)} < +\infty\right\}$
- Discrete convolution: $(h * a)[\boldsymbol{k}] = (a * h)[\boldsymbol{k}] = \sum_{\boldsymbol{l}\in\mathbb{Z}^d} a[\boldsymbol{l}]\, h[\boldsymbol{k} - \boldsymbol{l}]$

## Discrete-space Fourier transform

- Exponent notation: $\mathbf{z}^{-\boldsymbol{k}} = z_1^{-k_1} z_2^{-k_2} \cdots z_d^{-k_d}$
  with $\mathbf{z} = (z_1, \cdots, z_d) \in \mathbb{C}^d$ and $\boldsymbol{k} = (k_1, \cdots, k_d) \in \mathbb{Z}^d$
- $z$-transform: $A(\mathbf{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\, \mathbf{z}^{-\boldsymbol{k}}$
- Complex sinusoids: $e^{j\langle\boldsymbol{\omega},\boldsymbol{k}\rangle} = e^{j(\omega_1 k_1 + \omega_2 k_2 \cdots + \omega_d k_d)} = e^{j\omega_1 k_1} e^{j\omega_2 k_2} \cdots e^{j\omega_d k_d}$
- Discrete-space Fourier transform: $A(e^{j\boldsymbol{\omega}}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\, e^{-j\langle\boldsymbol{\omega},\boldsymbol{k}\rangle}, \quad \boldsymbol{\omega} \in \mathbb{R}^d$
- Inverse Fourier transform: $a[\boldsymbol{k}] = \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega}\in[-\pi,\pi]^d} A(e^{j\boldsymbol{\omega}}) e^{j\langle\boldsymbol{\omega},\boldsymbol{k}\rangle} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d$

## Useful properties

- Periodicity of DFT: $\forall \boldsymbol{k} \in \mathbb{Z}^d,\ A(e^{j(\boldsymbol{\omega}+2\pi\boldsymbol{k})}) = A(e^{j\boldsymbol{\omega}})$
- Parseval identity: $\langle a, b\rangle_{\ell_2(\mathbb{Z}^d)} = \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega}\in[-\pi,\pi]^d} A(e^{j\boldsymbol{\omega}}) B^*(e^{j\boldsymbol{\omega}}) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d$
- Young inequality: $\|(h * a)\|_{\ell_p(\mathbb{Z}^d)} \le \|h\|_{\ell_1(\mathbb{Z}^d)} \cdot \|a\|_{\ell_p(\mathbb{Z}^d)}$
- Wiener lemma: Let $h_{\mathrm{inv}} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{H(e^{j\boldsymbol{\omega}})}$ be the convolution inverse of $h \in \ell_1(\mathbb{Z}^d)$.
  Then, $h_{\mathrm{inv}} \in \ell_1(\mathbb{Z}^d)$ if $H(e^{j\boldsymbol{\omega}}) \neq 0,\ \forall \boldsymbol{\omega} \in [-\pi, \pi]^d$.

# Continuous-domain multidimensional signals

## Notations and definitions

- Continuous-domain signal (round brakets): $f(\boldsymbol{x})$ with $\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$
- $L_p$-norm: $\|f\|_{L_p(\mathbb{R}^d)} = \left(\int_{\boldsymbol{x}\in\mathbb{R}^d} |f(\boldsymbol{x})|^p \mathrm{d}x_1 \cdots \mathrm{d}x_d\right)^{\frac{1}{p}}$, for $1 \le p < +\infty$
- $L_2$-inner product: $\langle f, g\rangle_{L_2(\mathbb{R}^d)} = \int_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x})\, g^*(\boldsymbol{x}) \mathrm{d}x_1 \cdots \mathrm{d}x_d$
- Signal energy: $\|f\|^2_{L_2(\mathbb{R}^d)} \stackrel{\Delta}{=} \langle f, f\rangle_{L_2(\mathbb{R}^d)} = \int_{\boldsymbol{x}\in\mathbb{R}^d} |f(\boldsymbol{x})|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_d$
- Space of finite-energy functions: $L_2(\mathbb{R}^d) = \left\{f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^d, \|f\|^2_{L_2(\mathbb{R}^d)} < +\infty\right\}$
- Convolution: $(h * f)(\boldsymbol{x}) = (f * h)(\boldsymbol{x}) = \int_{\mathbf{y}\in\mathbb{R}^d} f(\mathbf{y})\, h(\boldsymbol{x} - \mathbf{y}) \mathrm{d}y_1 \cdots \mathrm{d}y_d$
- Linear shift-invariant operator: $\mathrm{L}\{f\}(\boldsymbol{x}) = \left(\mathrm{L}\{\delta\} * f\right)(\boldsymbol{x}) \stackrel{\mathcal{F}}{\longleftrightarrow} \hat{L}(\boldsymbol{\omega}) \cdot \hat{f}(\boldsymbol{\omega})$

## Fourier transform

- Frequency variable: $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_d) \in \mathbb{R}^d$
- Complex sinusoids (plane waves): $e^{j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle} = e^{j(\omega_1 x_1 + \omega_2 x_2 \cdots + \omega_d x_d)}$
- Fourier transform: $\hat{f}(\boldsymbol{\omega}) = \int_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x})\, e^{-j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle} \mathrm{d}x_1 \cdots \mathrm{d}x_d$
- Inverse Fourier transform: $f(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega}\in\mathbb{R}^d} \hat{f}(\boldsymbol{\omega}) e^{j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d$ (a.e.)

## Useful properties

- Parseval identity: $\langle f, g\rangle_{L_2(\mathbb{R}^d)} = \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega}\in\mathbb{R}^d} \hat{f}(\boldsymbol{\omega}) \hat{g}^*(\boldsymbol{\omega}) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d$
- Young inequality: $\|(h * f)\|_{L_p(\mathbb{R}^d)} \le \|h\|_{L_1(\mathbb{R}^d)} \cdot \|f\|_{L_p(\mathbb{R}^d)}$

# Sampling and reconstruction

## Notations and definitions

- Necessary condition for sampling: $f(\boldsymbol{x})$ is pointwise-continuous
- Discrete representation: $f[\boldsymbol{k}] = f(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{k}},\ \boldsymbol{k} \in \mathbb{Z}^d$
- Continuous-domain representation:

$$f_\delta(\boldsymbol{x}) = f(\boldsymbol{x}) \cdot \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \delta(\boldsymbol{x}-\boldsymbol{k}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} f[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$$

- Continuous-discrete signal reconstruction formula:

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad C(e^{j\boldsymbol{\omega}}) \cdot \hat{\varphi}(\boldsymbol{\omega})$$

Explicit Fourier calculation:

$$\hat{g}(\boldsymbol{\omega}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\hat{\varphi}(\boldsymbol{\omega})e^{-j\langle\boldsymbol{\omega},\boldsymbol{k}\rangle} = \hat{\varphi}(\boldsymbol{\omega}) \cdot \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]e^{-j\langle\boldsymbol{\omega},\boldsymbol{k}\rangle}$$

## Sampling and Poisson summation formula

- Fourier-domain effect of sampling

$$F(e^{j\boldsymbol{\omega}}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} f[\boldsymbol{k}]e^{-j\langle\boldsymbol{\omega},\boldsymbol{k}\rangle} = \hat{f}_\delta(\boldsymbol{\omega}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{\omega}+2\pi\mathbf{n})$$

(Hypothesis: $f[\boldsymbol{k}] \in \ell_1(\mathbb{Z}^d)$ and right-hand sum converging pointwise)

- Smoothness condition for finite-energy sampling:

$$f(\boldsymbol{x}) \in W_2^s(\mathbb{R}^d) \text{ with } s > \tfrac{d}{2} \quad \Rightarrow \quad \begin{cases} f[\boldsymbol{k}] \in \ell_2(\mathbb{Z}^d), \\ F(e^{j\boldsymbol{\omega}}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{\omega}+2\pi\mathbf{n}) \quad \text{a.e.} \end{cases}$$

- Sobolev's space of order $s$ ("$s$" derivatives in $L_2$-sense)

$$W_2^s(\mathbb{R}^d) = \left\{ f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^d : \int_{\boldsymbol{\omega}\in\mathbb{R}^d} (1+\|\boldsymbol{\omega}\|^{2s})|\hat{f}(\boldsymbol{\omega})|^2 \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d < +\infty \right\}$$